Framework for Mathematical Proficiency for Teaching

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Secondary school mathematics comprises far more than facts, routines, and strategies. It includes a vast array of interrelated mathematical concepts, ways to represent and communicate those concepts, and tools for solving all kinds of mathematical problems. It requires reasoning and creativity, providing learners with mathematical knowledge while also laying a foundation for further studies in mathematics and other disciplines.

To facilitate the learning of secondary school mathematics, teachers need a particular kind of proficiency. Mathematical proficiency for teaching at the secondary level is the mathematical expertise and skill a teacher has and uses for the purpose of promoting in students the understanding of, proficiency with, and appreciation for mathematics. It requires that teachers know not only more mathematics than they teach but also know it more deeply.

Mathematical proficiency for teaching (MPT) is unique to the work of teaching. It is different from the mathematical proficiency needed for engineering, accounting, or the medical professions. It is even different from the mathematical proficiency a mathematician needs. For example, a mathematician may prove a theorem and an architect may perform geometric calculations. For these users of mathematics, it is sufficient that they have the skills and understanding for the task at hand. But a teacher's work includes these tasks as well as interpreting students' mathematics, developing multiple representations of a mathematical concept, knowing where students are on the path of mathematical understanding, and so on.

Mathematical proficiency for teaching is *dynamic*. We make a distinction between knowledge and proficiency. Knowledge may be seen as static and something that cannot be directly observed, whereas proficiency can be viewed as the dynamic use of the knowledge one has. Proficiency can be observed in a teacher's actions and the decisions he or she makes. Also, because of its dynamic nature, MPT grows and deepens in the course of a teacher's career.

The focus of our MPT framework is on *secondary school* mathematics. That is, we seek to characterize the mathematical proficiency that is useful to secondary teachers as distinct from the proficiency needed by elementary school mathematics teachers. We believe that MPT for secondary school is different from MPT for elementary school in at least four ways: (1) There is a wider range of mathematics content (i.e., more topics are studied); (2) There is a greater emphasis on formality, axiomatic systems, and rigor in regard to mathematical proof; (3) There is more explicit attention to mathematical structure and abstraction (e.g., identities, inverses, domain, and undefined elements); and (4) The cognitive development of secondary students is such that they can reason differently from elementary school children about such matters as proportionality, probability, and mathematical induction.

Our framework has been developed out of *classroom practice* and we have drawn examples from a wide variety of classroom contexts. We have examined episodes occurring in the work of prospective and practicing secondary mathematics teachers and mathematics educators at the college level. From this collection, we have determined elements of mathematics proficiency that would be beneficial to secondary mathematics teachers. We describe a wide sample, as opposed to a comprehensive catalog, of mathematical proficiency for teaching that comes from our analyses of these classroom episodes.

Mathematical proficiency for teaching is not the same as proficiency in pedagogy. Being equipped with the proficiency described in our MPT framework is not simply a matter of "knowing the mathematics" plus "knowing how to teach." The task of teaching mathematics cannot be partitioned into such simple categories.

A Framework for MPT

Mathematical proficiency for teaching (MPT) can be viewed from three perspectives or through three lenses: mathematical proficiency, mathematical activity, and mathematical work of teaching (Figure 1). Each perspective provides a different view of MPT. MPT is a developing quality and not an endpoint.

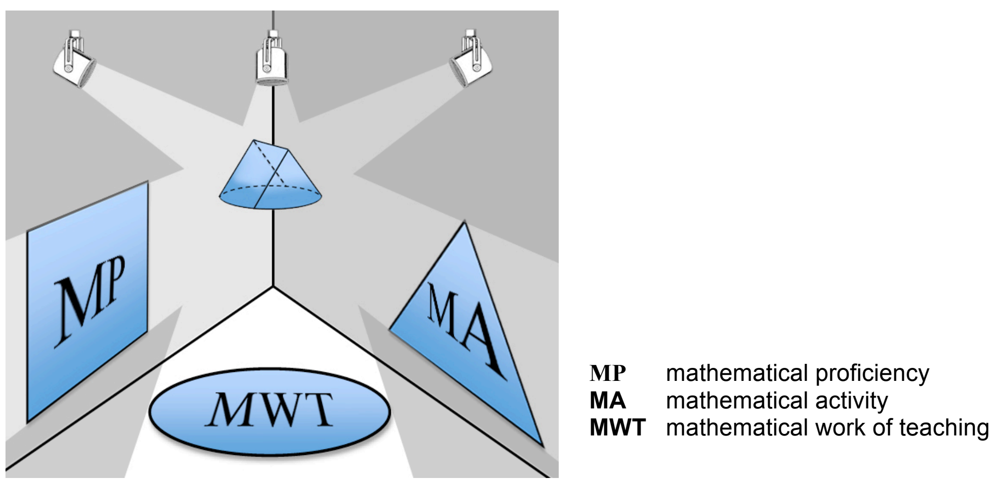

Figure 1. Mathematical proficiency for teaching viewed from three perspectives.

Mathematical proficiency includes aspects of mathematical knowledge and ability, such as conceptual understanding and procedural fluency, that teachers need themselves and that they seek to foster in their students. The mathematical proficiency teachers need, however, goes well beyond what one might find in secondary students. The students' development of mathematical proficiency usually depends heavily on how well developed the teacher's proficiency is.

Proficiency in mathematical activity can be thought of as "doing mathematics." Examples include representing mathematical objects and operations, connecting mathematical concepts, modeling mathematical phenomena, and justifying mathematical arguments. This facet of teachers' mathematical proficiency is on display as they engage students in the day-to-day study of mathematics. Teachers need deep knowledge, for example, of what characterizes the structure of mathematics (as opposed to conventions that have been adopted over the centuries) and how to generalize mathematical findings. The more a teacher's proficiency in mathematical activity has developed, the better equipped he or she will be to facilitate the learning and doing of mathematics.

Proficiency in the mathematical work of teaching diverges sharply from the mathematical proficiency needed in other professions requiring mathematics. One of its aspects is an understanding of the mathematical thinking of students, which may include, for example, recognizing the mathematical nature of their errors and misconceptions. Another aspect of the mathematical work of teaching is knowledge of and proficiency in the mathematics that comes before and after what is being studied currently. A teacher benefits from knowing what students have learned in previous years so that he or she can help them build upon that prior knowledge. The teacher also needs to provide a foundation for the mathematics they will be learning later, which requires knowing and understanding the mathematics in the rest of the curriculum.

The three components of MPT—mathematical proficiency, mathematical activity, and mathematical work of teaching—together form a full picture of the mathematics required of a teacher of secondary mathematics. It is not enough to know the mathematics that students are learning. Teachers must also possess a depth and extent of mathematical proficiency that will equip them to foster their students' mathematical proficiency. Mathematical proficiency informs the other two perspectives on MPT: Mathematical activity and the mathematical work of teaching emerge from, and depend upon, the teacher's mathematical proficiency.

An Example of MPT Use

In responding to the following situation, no matter how it is handled pedagogically, the teacher needs to make use of all facets of his or her MPT:

> In an Algebra II class, students had just finished reviewing the rules for exponents. The teacher wrote $x^m \cdot x^n = x^5$ on the board and asked the students to make a list of values for m and n that made the statement true. After a few minutes, one student asked, "Can we write them all down? I keep thinking of more."

To decide whether the student's question is worth pursuing, frame additional questions appropriately, and know how to proceed from there, the teacher needs conceptual understanding and productive disposition (two aspects of mathematical proficiency). The concept of an exponent is more complicated than might be initially apparent. Does the rule $x^m \bullet x^n = x^{m+n}$ always apply? Must the domain of *x* be restricted? Must the domain of *m* and *n* be restricted? These are questions the teacher needs sufficient mathematical proficiency to address. With respect to mathematical activity, the teacher's proficiency in representing exponents, knowing constraints that may be helpful in dealing with them, and making connections between exponents and other mathematical phenomena are all crucial to successfully teaching the concept. What are the advantages of a graphical representation of an exponential function as opposed to a symbolic representation? How is the operation of exponentiation connected to the operation of multiplication? Does an exponent always indicate repeated multiplication? With respect to the mathematical work of teaching, it is critical that the teacher knows and understands the mathematics that typically comes before and after the point in the curriculum where a problem like the one involving the rule $x^m \bullet x^n$ is addressed. For example if this problem is being discussed in a beginning algebra course, it is important to realize that students have probably had limited exposure to exponents and may think about them only in terms of the repeated multiplication of natural numbers. And to lay a good foundation for later studies of exponential functions, the teacher needs to know that there may be discontinuity in the graph of x^n depending on the domain of both the base and the exponent.

Elaboration of the MPT Perspectives

The philosopher Gilbert Ryle (1949) claimed that there are two types of knowledge: The first is expressed as "knowing that," sometimes called *propositional* or *factual* knowledge, and the second as "knowing how," sometimes called *practical* knowledge. Because we wanted to capture this distinction and at the same time to enlarge the construct of *mathematical knowledge for teaching* to include such mathematical aspects as reasoning, problem solving, and disposition, we have adopted the term *proficiency* throughout this document instead of using *knowledge*. We use *proficiency* in much the same way as it is used in *Adding It Up* (Kilpatrick, Swafford, & Findell, 2001) except that we are applying it to teachers rather than students and to their teaching as well as to their knowing and doing of mathematics.

An outline of our framework for the three perspectives on MPT is shown in Figure 2. In this section, we amplify each perspective in turn.

1. **Mathematical proficiency**
    - Conceptual understanding
    - Procedural fluency
    - Strategic competence
    - Adaptive reasoning
    - Productive disposition
    - Historical and cultural knowledge

2. **Mathematical activity**
    - Mathematical noticing
        - Structure of mathematical systems
        - Symbolic form
        - Form of an argument
        - Connect within and outside mathematics
    - Mathematical reasoning
        - Justifying/proving
        - Reasoning when conjecturing and generalizing
        - Constraining and extending
    - Mathematical creating
        - Representing
        - Defining
        - Modifying/transforming/manipulating
    - Integrating strands of mathematical activity

3. **Mathematical work of teaching**
    - Probe mathematical ideas
    - Access and understand the mathematical thinking of learners
    - Know and use the curriculum
    - Assess the mathematical knowledge of learners
    - Reflect on the mathematics of practice

Figure 2. Framework for mathematical proficiency for teaching (MPT).

Mathematical Proficiency

The principal goal of secondary school mathematics is to develop all facets of the learners' mathematical proficiency, and the teacher of secondary mathematics needs to be able to help students with that development. Such proficiency on the teacher's part requires that the teacher not only understand the substance of secondary school mathematics deeply and thoroughly but also know how to guide students toward greater proficiency in mathematics. We have divided the teacher's mathematical proficiency into six strands, shown in Figure 2, to capture the multifaceted nature of that proficiency.

There is a range of proficiency in each strand, and a teacher may become increasingly proficient in the course of his or her career. At the same time, certain categories may involve greater depth of mathematical knowledge than others. For example, *conceptual understanding* involves a different kind of knowledge than *procedural fluency,* though both are important. Only rote knowledge is required in order to demonstrate procedural fluency in mathematics. Conceptual understanding, however, involves (among other things) knowing *why* the procedures work.

Conceptual Understanding

Conceptual understanding is sometimes described as the "knowing *why*" of mathematical proficiency. A person may demonstrate conceptual understanding by such actions as deriving needed formulas without simply retrieving them from memory, evaluating an answer for reasonableness and correctness, understanding connections in mathematics, or formulating a proof.

Some examples of conceptual understanding are the following:

1. Knowing and understanding where the quadratic formula comes from (including being able to derive it),
2. Seeing the connections between right triangle trigonometry and the graphs of trig functions, and
3. Understanding how the introduction of an outlying data point can affect mean and median differently.

Procedural Fluency

A person with procedural fluency knows some conditions for when and how a procedure may be applied and can apply it competently. Procedural fluency alone, however, would not allow one to independently derive new uses for a previously learned procedure, such as completing the square to solve $ax^6 + bx^3 = c$. Procedural fluency can be thought of as part of the "knowing *how*" of mathematical proficiency. Such fluency is useful because the ability to quickly recall and accurately execute procedures significantly aids in the solution of mathematical problems.

The following are examples of procedural fluency:
1. Recalling and using the algorithm for long division of polynomials,
2. Sketching the graph of a linear function,
3. Finding the area of a polygon using a formula, and
4. Using key words to translate the relevant information in a word problem into an algebraic expression.

Strategic Competence

Strategic competence requires procedural fluency as well as a certain level of conceptual understanding. Demonstrating strategic competence requires the ability to generate, evaluate, and implement problem-solving strategies. That is, a person must first be able to generate possible problem-solving strategies (such as utilizing a known formula, deriving a new formula, solving a simpler problem, trying extreme cases, or graphing), and then must evaluate the relative effectiveness of those strategies. The person must then accurately implement the chosen strategy. Strategic competence could be described as "knowing how," but it is different from procedural fluency in that it requires creativity and flexibility because problem-solving strategies cannot be reduced to mere procedures.

Examples of strategic competence are the following:
1. Recognizing problems in which the quadratic formula is useful (which goes beyond simply recognizing a quadratic equation or function), and
2. Figuring out how to partition a variety of polygons into "helpful" pieces so as to find their areas.

Adaptive Reasoning

A teacher or student with adaptive reasoning is able to recognize current assumptions and adjust to changes in assumptions and conventions. Adjusting to these changes involves comparing assumptions and working in a variety of mathematical systems. For example, since they are based on different assumptions, Euclidean and spherical geometries are structurally different. A person with adaptive reasoning, when introduced to spherical geometry, would consider the possibility that the interior angles of a triangle do not sum to 180°. Furthermore, he or she would be able to construct an example of a triangle, within the assumptions of spherical geometry, whose interior angles sum to more than 180°.

Adaptive reasoning includes the ability to reason both formally and informally. Some examples of formal reasoning are using rules of logic (necessary and sufficient conditions, syllogisms, etc.) and structures of proof (by contradiction, induction, etc.). Informal reasoning may include creating and understanding appropriate analogies, utilizing semi-rigorous justification, and reasoning from representations.

Examples of adaptive reasoning are as follows:
1. Recognizing that division by an unknown is problematic,
2. Working with both common definitions for a trapezoid,

3. Operating in more than one coordinate system,
4. Proving an if-then statement by proving its contrapositive, and
5. Determining the validity of a proposed analogy.

Productive Disposition

Those people with a productive disposition believe they can benefit from engaging in mathematical activity and are confident that they can succeed in mathematical endeavors. They are curious and enthusiastic about mathematics and are therefore motivated to see a problem through to its conclusion, even if that involves thinking about the problem for an extended time so as to make progress. People with a productive disposition are able to notice mathematics in the world around them and apply mathematical principles to situations outside the mathematics classroom. They possess Cuoco's (1996) "habits of mind."

Examples of productive disposition are as follows:

1. Noticing symmetry in the natural world,
2. Persevering through multiple attempts to solve a problem, and
3. Taking time to write and solve a system of equations for comparing phone service plans.

Historical and Cultural Knowledge

Having knowledge about the history of mathematics is beneficial for many reasons. One prominent benefit is that a person with such knowledge will likely have a deeper understanding of the origin and significance of various mathematical conventions, which in turn may increase his or her conceptual understanding of mathematical ideas. For example, knowing that the integral symbol ∫ is an elongated *s*, from the Latin *summa* (meaning *sum* or *total*) may provide a person with insight about what the integral function is. Some other benefits of historical knowledge include an awareness of which mathematical ideas have proven the most useful in the past, an increased ability to predict which mathematical ideas will likely be of use to students in the future, and an appreciation for current developments in mathematics.

Cross-cultural knowledge (i.e., awareness of how people in various cultures or even in various disciplines conceptualize and express mathematical ideas) may have a direct impact on a person's mathematical understanding. For example, a teacher or student may be used to defining a rectangle in terms of its sides and angles, but people in some non-Western cultures define a rectangle in terms of its diagonals. Being able to conceptualize both definitions can strengthen one's mathematical proficiency.

The following are additional examples of historical and cultural knowledge:

1. Being familiar with the historic progression from Euclidean geometry to multiple geometric systems,
2. Being able to compare mathematicians' convention of measuring angles counterclockwise from horizontal with the convention (used by pilots, ship captains, etc.) of indicating directions in terms of degrees clockwise from North,

3. Understanding similarities and differences in algorithms typically taught in North America and those taught elsewhere,
4. Knowing that long-standing “open problems” in mathematics continue to be solved and new problems posed, and
5. Recognizing the increasing use of statistics in the business world.

Mathematical Activity

Through a mathematical activity perspective, we acknowledge that mathematical knowledge has a dynamic aspect by describing mathematical actions. The categories in the dimension of mathematical activity organize the verbs of doing mathematics—the actions one uses with mathematical objects. The three strands—mathematical noticing, mathematical reasoning, and mathematical creating—intertwine in mathematical activity.

Fundamental to each of the three strands is constraining, extending or otherwise altering of conditions and forms. To constrain in mathematics means to define the limits of a particular mathematical idea. Constraints can be removed, altered, or replaced to explore the resulting new mathematics[1]. Mathematical relationships and properties can be tested for extended sets of numbers. As Cuoco, Goldenberg and Mark (1996) argue, “mathematicians talk small and think big” (p. 384). Teachers need to move flexibly between related small and big ideas. They use constraining, extending and altering as ways to refine ideas to create valid statements from intuitive notions and observations.

Mathematical noticing

Our first category of mathematical activity, mathematical noticing[2], involves recognizing similarities and differences in structure, form, and argumentation both in mathematical settings and in real world settings. Mathematical noticing requires identifying mathematical characteristics that are particularly salient for the purpose at hand and focusing in on those characteristics in the presence of other available candidates for foci.

Structure of mathematical systems

Structure is the essence of mathematics, and noticing structure is foundational to making mathematical conclusions. The structure on which one focuses can be the definitions and axioms that govern a mathematical system, the form of a symbolic expression, or the organization of a mathematical argument. Noticing and using the structure of mathematical systems underpins other mathematical activities such as deriving properties of a system. Noticing features of a symbolic form is fundamental to knowing what moves are legitimate in transforming those forms into more useful

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[1] This text seemed to fit here, but it bothers me not to have something similar for extend or alter. Do we *need* this comment on constrain?

[2] Although our notion of “mathematical noticing” may have some features in common with Goodwin’s “professional noticing” [Goodwin,C. (Sept., 1994). American Anthropologist, New Series, 96(3), 606-633.] the terms are not equivalent nor is one derived from the other.

configurations. Noticing the structure or organization of a mathematical argument is essential to evaluating the validity of that argument. Whereas many users of mathematics perform using these system, form, and argumentation structures, teachers need to notice similarities and differences among these structures in varied mathematical settings.

At the secondary level, the rate of introduction of new mathematical systems increases, and although the new systems use similar operations on similar objects, teachers need to be constantly vigil regarding the constraints under which each system operates. For example, not all properties of multiplication applied to real numbers generalize to multiplication applied to matrices. With the increasing rate of introduction of new systems at the secondary level, there is a more pronounced need to notice and account for differences in structure of different mathematical systems. Teachers need to notice invariant as well as changing aspects of mathematical structure as the curriculum moves from the study of rational numbers to the study of real and complex numbers, variables, polynomials, matrices, and functions. As new entities such as the inverse function and composition of functions are introduced, teachers need to recognize the entities of inverses and compositions across a broad range of mathematical settings ranging from $\Re$-to-$\Re$ functions in algebraic settings to transformations in geometric settings.

Examples of noticing mathematical structure are:

1. Noticing the effects on a geometry when the parallel postulate is not assumed;
2. Noticing connections between (features of) representations of the same mathematical object or different methods for solving problems (e.g., notice the structural similarities in the Euclidean algorithm and the long division algorithm);
3. Noticing differences between mathematical objects of different classes (e.g., notice that the properties of a set are not inherited when the set is extended to a superset, and that properties of a set can be restricted or expanded when some elements of the set are eliminated);
4. Noticing differences between the same objects in different systems (e.g., notice the difference in solutions when solving an equation in the real number system and in the complex number system).
5. Noticing differences and similarities in notation and distinguish among the meanings of notations that are similar in appearance (e.g, notice differences in the uses of familiar notation such as the exponent -1 (e.g., x^{-1} and f^{-1}) depending on context, and to be able to identify and explain the conditions under which particular meanings for the notation are appropriate).
6. Noticing differences in algebraic structure (e.g., notice properties of a system such as the field properties, properties of equivalence relations, and properties of equality) and apply the knowledge of this structure to algebraic transformations.[3]

[3] Algebraic transformations such as the production of equivalent expressions and equivalent equations are the core of many school algebra courses.

7. Notice similarities in algebraic structure (e.g., recognizing the similarity of the structure of switching circuits, conjunction and disjunction of sets, and Boolean algebra).

Symbolic form

Recognizing algebraic symbolic forms allows teachers to identify and use potential symbolic rules with those forms.

Examples of noticing symbolic forms are:

1. Being aware that that the truth of $f(a) + f(b) = f(a + b)$ depends on the nature of the function f, and that students tend to apply this “student’s distributive property” indiscriminately. Teachers with this awareness can alert their students to the need to be wary of overgeneralization of this linearity property and test the validity of each potential rule.
2. Being aware that familiar operations do not have the same meaning when applied to different mathematical objects and structures, and hence knowing not to generalize properties of multiplication over the set of real numbers to multiplication over the set of matrices.
3. Being cautious of extending the rules of exponents, developed and proved for natural number exponents, to negative, rational, real, or complex exponents.

Form of an argument

Secondary teachers have a particular need to notice the form of mathematical arguments, whether advanced in a textbook or by a student. Noticing the form of a mathematical argument allows teachers to identify missing elements of the argument or redundant portions of the argument.

Connect within and outside mathematics

Connecting within mathematics requires teachers to extract the characteristics and structure of the mathematics they are teaching and notice those characteristics and structure in other areas of mathematics. Teachers who notice connections between mathematical representations of the same entity and between mathematical entities and their properties can provide rich and challenging environments for their students when they are able to move smoothly from question to question, both fielding student questions and posing challenges that require students to connect mathematical ideas.

Examples of noticing connections within mathematics are:

1. Noticing different manifestations and representations of the same mathematical system (e.g., notice that paper-folding, symmetries of a triangle, paper-and-pencil games, and Escher-type drawings are all venues for studying transformations such as reflections, rotations, translations, and glide reflections, and that transformations can be represented and

manipulated through matrix operations as well as through mappings on a plane).

2. Recognizing relationships between alternative algorithms, student-generated algorithms, and standard algorithms (e.g., noticing that Peasant multiplication and standard multiplication algorithms can be derived using the field properties).
3. Noticing the affordances of the different representations and that different representations highlight different strengths and weaknesses of what is being represented.

Connecting to areas outside of mathematics requires teachers to have a disposition to notice mathematics outside of their classroom, both within and beyond the boundaries of the school walls and to seek mathematical explanations for real world quantitative relationships. Teachers who introduce ideas such as these to students can not only engage them in simulating what those who use mathematics in real life do but also help them to understand the mathematical ideas shaping the world in which they live. The point is not that there is not some ordained list of applications that a teachers needs to know, but rather that there are intriguing topics that teachers can explore with their students by applying mathematics at a secondary level and that teachers should be willing and able to seek out the resources to investigate these topics. Connecting within and outside mathematics means looking for and noticing applications of mathematics as well as situations from which to extract mathematics. The secondary teacher needs to notice not only that a connection exists but exactly how that connection relates to the mathematics they are teaching. Every teacher may not need to know something about a particular connection, but all teachers need to know the properties of the mathematical entities about which they are teaching well enough to recognize an application when they see it. This recognition involves seeing the properties of the mathematical entities well enough to match them to the situation (Zbiek & Conner, 2006).

Teaching secondary students about mathematics in real-world settings requires that teachers recognize the properties of mathematical entities in such situations. They should be able to see a property of a mathematical entity in a situation, they should notice characteristics of situations that they can associate with a mathematical entity, and they should be able to recognize the constraints that the context places on a mathematical result.

Examples of noticing connections to the world outside of mathematics are:

1. Noticing the mathematics that underpins today's electronic technology (e.g., noticing that video games employ matrix operations to animate images on the screen through geometric transformations)
2. Noticing ways that mathematics underpins different industries (e.g., noticing that designers of automobiles use Bezier curves to render pictures of new designs for cars).

Mathematical reasoning

Our second category of mathematical activity is mathematical reasoning. Mathematical reasoning includes justifying and proving as well as reasoning in the context of conjecturing and generalizing. Mathematical reasoning results in the production of a mathematical argument or a rationale that supports the plausibility of a conjecture or generalization.

Justifying/proving

Teaching mathematics well requires justifying mathematical claims through logically deduced connections among mathematical ideas. Formal justification, or proof, requires basing arguments on a logical sequence of statements supported by definitions, axioms, and known theorems, whereas informal arguments involve reasoning from empirically derived—but often limited—data, reasoning by analogy, establishing plausibility based on similar instances, and the like.

Teachers of secondary mathematics need a different sort of justification ability from that of other users of mathematics because they are required to formulate and structure arguments at a range of appropriate levels. Teachers need to be comfortable with a range of strategies for mathematical justification, including both formal justification and informal arguments. Secondary mathematics teachers need to be able to understand and formulate different levels and types of mathematically and pedagogically viable justifications and proofs (e.g., proof by contradiction, proof by induction). They are called on to consider or generate empirical evidence, formulate conjectures from that evidence, and prove or disprove those conjectures deductively. They also need to recognize the need to specify assumptions in an argument, and they must be able to state assumptions on which a valid mathematical argument depends. Teachers' arguments often need not be as elegant as those for which mathematicians typically strive, and teachers need to be able to create proofs that explain as well as proofs that convince (Hersch, 1993).

When creating formal or informal arguments, teachers need to be on the alert for special cases (e.g., any number raised to the zero power is not always 1), they need to recognize or generate an exhaustive list of cases, and they need to recognize the limitations of reasoning from diagrams. Not only do they need experience with justification and proof but they also need to be able to craft explanations that communicate aspects of justification at an appropriate level. For example, they need to be able to explain why a process does not generalize when applied to a different entity, and they need to be able to discern the logic or organizing idea of a formal proof so they can explain it to students who have limited experience in constructing proofs.

Examples arising from our situations include:

1. Constructing an array of explanations for why the sum of the first n natural numbers is $\frac{n(n+1)}{2}$, including appealing to cases, by making strategic choices for pair-wise grouping of numbers, and by appealing to arithmetic sequences and properties of such sequences.
2. Arguing by contradiction (excluded middle): To prove that if the opposite angles of a quadrilateral are supplementary, then the quadrilateral can be inscribed in a circle, one can construct a circumcircle about three vertices of a quadrilateral and argue that if the fourth vertex can be in neither the interior nor the exterior of the circle, then the fourth vertex must be on the circumcircle, and therefore the quadrilateral can be inscribed in a circle.

Reasoning when conjecturing and generalizing

Finding theorems to prove is a typical mathematical activity in which mathematicians engage. In school mathematics, students (and teachers) engage in a similar activity when they develop conjectures based on their observations and data they have generated. When students formulate a conjecture, teachers need to evaluate students' rationale for that conjecture, and in some cases produce an argument that convinces students of its plausibility. Given a plausible conjecture—generated by the teacher or generated by students—a teacher must be able to test the conjecture with different domains or sets of objects. Students often state conjectures (or generalizations) in an overly broad way, so their teachers need to be adept at constraining the domain of a proposed generalizations to produce a statement that is true.

Generalizing is the act of extending the domain to which a set of properties apply from multiple instances of a class or from a subclass to a larger class of mathematical entities, thus identifying a larger set of instances to which the set of properties applies. When generalizing, students may develop a formal argument that establishes the generalization as being true, and that argument must be evaluated by the teacher. In other instances, the teacher needs to produce an argument that convinces students of a generalization's truth or explains some aspect related to the statement or its domain of applicability.

Teachers engage in mathematical reasoning in the context of conjecturing and generalizing when they consider the effects of constraining or extending the domain, argument, or class of objects for which a mathematical statement is or remains valid while preserving the structure of the mathematical statement (e.g., extending the concept of absolute value to a modulus definition as the domain is extended from real to complex numbers; extending the object "triangle" from Euclidean to spherical geometry). They need to recognize when it is useful to relax or constrain mathematical conditions (e.g., recognizing that it is not true that any number raised to the 0th power is equivalent to 1). They make mathematical generalizations by extending the domain within which a set of properties applies, thus identifying a larger set of instances to which the properties apply. The mathematical reasoning involved in constraining and extending enables teachers to create extensions to given problems and questions.

Teachers also engage in mathematical reasoning in the context of conjecturing and generalizing when they create and use counterinstances of generalizations. The creation of counterinstances requires one to reason about the domain of applicability of a generalization and what results when that domain is constrained or extended. For example, the generalization that multiplication is commutative can be shown to be false when considering matrix multiplication.

Constraining and extending

With secondary mathematics as the bridge between prealgebra mathematics and collegiate mathematics, secondary mathematics teachers are often challenged to explore the consequences of imposing or relaxing constraints. To *constrain* in mathematics means to define the limits of a particular mathematical idea. When finding the inverses of a function, one must sometimes constrain the domain if one wants the inverse to be a function as well. The inverse of $f(x) = \sin x$ is a function only if the new domain is restricted. Constraints can be removed or replaced to explore the resulting new mathematics. When mathematicians tinkered with the constraint of Euclid's fifth postulate, new geometries were formed. When one removes the constraint of the plane in using Euclidean figures, the mathematics being used changes as well. Secondary mathematics teachers regularly encounter situations in which to provide a suitable response, they must tailor a generalization so that it can reasonably be extended to a larger domain of applicability. For example, they may have to analyze the extent to which properties of exponents generalize from natural number exponents to rational number or real number exponents, or they may find it useful to generate an example of a situation for which multiplication is not commutative. Teachers with an understanding of the mathematics their students will encounter in further coursework can structure arguments so that they extend to a more general case. For example, teachers who recognize that a graphical approach to solving polynomial equations is far more generalizable than the usual set of polynomial factoring techniques may tend to provide their students with a more useful technique.

Mathematical creating (producing a new mathematical entity)

The essence of mathematical creating is the production of new mathematical entities through the mathematical activities of representing, defining, generalizing and transforming.

Representing (new way to convey)

Inherent in mathematical work is the need to represent mathematical entities in ways that reflect given structures, constraints, or properties. The creation of representations is particularly useful in creating and communicating examples, nonexamples, and counterexamples for mathematical objects, generalizations, or relationships. Teachers need to be fluent in the rapid construction of representations that underscore key features of the represented entity.

The resulting representations take many different forms. Each representation affords different views of the mathematical object, but several different representations can highlight the same feature. Teachers need to be able to assess what features of the mathematical entity each form captures and what features it obscures. Different forms might be particularly helpful in answering certain types of questions or solving particular types of problems. Representing involves choosing or creating a useful form that conveys the crucial aspects of the mathematical entity that are needed for the task at hand.

Some types of mathematical representations are common. Teachers need to be able to create representations of these common forms in ways that reflect these conventions. Teachers also need to be able to create representations of less common and even novel forms. In this activity, attention to structures, constraints and properties is critical.

Defining (new object)

The mathematical activity of defining is the creation of a new mathematical entity by specifying its properties. Generating a definition requires identifying and articulating a combination of a set of characteristics and the relationships among these characteristics in such a way that the combination can be used to determine whether an object, action, or idea belongs to a class of objects, actions, or ideas. Constraining or extending the class of objects under consideration is one way the need for a new definition arises.

Teachers of secondary mathematics use definitions in their daily work. They need to be able to appeal to a definition to resolve mathematical questions, and they need to be able to reason from a definition. Less frequently, teachers need to create definitions and to assess the definitions that students create or propose.

Modifying/Transforming/Manipulating (new form)

Transforming is the activity of moving from one mathematical entity to another though constraining, extending or altering conditions or manipulating representations[4]. Perhaps the most recognizable form of transforming is symbolic manipulation. Teachers need to see these transformations as purposeful activities undertaken to produce a symbolic form that conveys particular information. Transformations of graphs (e.g., window changes, bin sizes) to create more meaningful representations are similarly important. Translations, or transformations between types of representations, are not only the goal of some tasks but also are problem-solving tools. Whether technology supported or completed by hand, transforming one representation to another representation (of a similar or different form) is fundamental in solving problems. With access to a broad range of representational forms, teachers can use or create equivalent representations to reveal different information or to foreground a particular concept.

Integrating strands of mathematical activity

Mathematical modeling provides one example of how the strands of mathematical activity intertwine in mathematical work. A popular description of the modeling process starts with a real-world problem that is translated into a formal mathematical system. Mathematical noticing occurs as the modeler specifies the conditions and assumptions that matter in the real-world setting. Devising the model requires mathematical creating informed by mathematical reasoning. After a potential model has been generated, mathematical creating takes over as the model is manipulated until a solution is found. The solution is mapped back to the real world to be tested with the problem through mathematical noticing. If a the real-world conclusion does not align with the modeling goal, aspects of the model, such as initial conditions that are assumed, could be constrained, expanded, or altered to form a new model. It is important to note that the issue is one of fit and utility rather than absolute correctness.

Mathematical modeling activities in secondary school might involve authentic modeling tasks that involve the generation of novel models or more restricted modeling work that is done in the service of students learning *curricular mathematics*, the mathematics that is the focus of classroom lessons (Zbiek & Conner, 2006). A close analysis of mathematical modeling across these contexts suggests that mathematical modeling is a nonlinear process that incorporates the three strands of mathematical activity (for an elaboration of modeling activities see Zbiek and Conner (2006)). Schoenfeld (1994) points out that the validity of the ensuing analysis depends on the accuracy of both of the mappings to and from the formal system, which seems to attest to the importance of mathematical noticing.

Mathematical Work of Teaching

The mathematical work of teaching requires that teachers be able to help someone else know and do mathematics. In Ryle's (1949) terminology, the mathematical work of teaching requires both knowing how and knowing that. It moves beyond the goal of establishing a substantial and continually growing proficiency in mathematics for oneself as a teacher to include the goal of effectively helping one's students develop mathematical proficiency.

Not only should teachers of secondary mathematics be able to know and do mathematics themselves, but also their proficiency in mathematics must prepare them to facilitate their students' development of mathematical proficiency. Possessing proficiency in the mathematical work of teaching mathematics enables teachers to integrate their knowledge of content and knowledge of processes to increase their students' mathematical understanding.

Analyze Mathematical Ideas

The first category of the mathematical work of teaching addresses the type of knowledge that is useful for investigating and pulling apart mathematical ideas. Mathematics is dense. One goal in doing mathematics is to compress numerous complex ideas into a few succinct, elegant expressions. These expressions can be used to build additional ideas that will also become compressed. Although mathematical efficiency and rigor are essential if one is to engage in complex mathematical thinking, they can also cause confusion, especially for those just being initiated into the culture of mathematics.

Analyzing mathematical ideas requires a broad knowledge of mathematical content and associated mathematical activities such as defining, representing, justifying, and connecting. Teachers need mathematical knowledge that will help them to pull apart mathematical ideas in ways that allow the ideas to be reassembled as students mature mathematically. They need to recognize and honor the conventions and structures of mathematics and recognize the complexity of elegant mathematical ideas that have been compressed into simple forms.

Examples of analyzing mathematical ideas are as follows:

1. Understanding the role of the domain in determining the values for which a function is defined and the role of operation in determining the inverse of an element of a set.
2. Recognizing the similarities and differences between multiplying real numbers and matrices.
3. Exploring the standard deviation of a set of data in terms of an average distance each value is from the mean of the set of data.
4. Exploring the various meanings of division—partitive and quotitive—to recognize that division is more than the inverse operation of multiplication.

Access and Understand the Mathematical Thinking of Students

The second category refers to knowledge that helps teachers understand how their students are thinking about mathematics. A proficient teacher uncovers students' mathematical ideas, seeing the mathematics from a learner's perspective. Teachers can gain some access to students' thinking through written work they do in class or at home, but much of that information is highly inferential. Through discourse with students about their mathematical ideas, the teacher can learn more about the thinking behind their written products. Classroom interactions play a significant role in teachers' understanding of what students know and are learning. It is through a particular kind and quality of discourse that implicit mathematical ideas are exposed and made more explicit.

Students often discuss mathematics using vague explanations or terms that have a colloquial meaning different from their mathematical meaning. A teacher needs the proficiency to interpret imprecise student explanations, help students focus on essential mathematical points, and help them learn conventional terms. Success in such endeavors requires understanding the nuances and implications of students' understanding and recognizing what is right about their thinking as well as features of their thinking that lead them to unproductive conceptions. Achieving such a balance requires the teacher to have an extensive knowledge of mathematical terminology, formal reasoning processes, and conventions, as well as an understanding of differences between colloquial uses and mathematical uses of terms.

Examples of mathematical knowledge needed for accessing students' mathematical knowledge are:

1. In a class discussion of Platonic solids, a student might propose a conjecture about the number of sides and number of vertices. Some students may interpret *sides* to mean *faces*; others may be thinking *edges*. A teacher who knows and anticipates such potential confusion can use it to motivate the class to reject the imprecise term *side* and define the terms *face* and *edge*. This elaboration can be handled without losing sight of the valuable conjecture made by the student.
2. When a student is struggling to express an idea algebraically, a teacher needs a collection of alternate, useful representations (e.g., graph, table, drawing, set of examples) that may help a student share important ideas.
3. Teachers' mathematical knowledge helps them design mathematical tasks that expose students thinking and maintain a high level of cognitive demand as the tasks are discussed.

Know and Use the Curriculum

The third category refers to the mathematical knowledge that helps teachers know and use the curriculum to help students connect mathematical ideas and progress to a deeper and better grounded mathematics. How mathematical knowledge is used to teach mathematics in a specific classroom or with a specific learner or specific group of learners is influenced by the curriculum that organizes the teaching and learning. A teacher's mathematical proficiency can help make that curriculum meaningful, connected, relevant, and useful. For example, a teacher who is proficient in the mathematical work of teaching may have a perspective on the curriculum for the concept of area that includes ideas about measure, descriptions of two-dimensional space, measures of space under a curve, measures of the surface of three-dimensional solids, infinite sums of discrete regions, operations on space and measures of space, foundations of the geometric properties of area, and useful applications involving area. This perspective on the curriculum is very different from that of someone who thinks of area only in terms of formulas for polygonal regions.

Proficiency in knowing and using the mathematics curriculum in teaching requires a teacher to identify foundational or prerequisite concepts that enhance the learning of a concept as well as how the concept being taught can serve as a foundational concept for future learning. The teacher needs to know how the concept fits each student's learning trajectory. The teacher also needs to be aware of common mathematical misconceptions and how those misconceptions may sometimes arise from instruction. Proficient mathematics teachers understand that there is not a fixed order for learning mathematics but rather multiple ways to approach a mathematical concept and to revisit it. Mathematical concepts and processes evolve in the learner's mind, becoming more complex and sophisticated with each iteration. Mathematical proficiency prepares a teacher to enact a curriculum that not only connects mathematical ideas explicitly but also develops a disposition in students so that they expect mathematical ideas to be connected and an intuition so that they see where those connections might be (Cuoco, 2001).

A teacher proficient in the mathematical work of teaching understands that a curriculum contains not only mathematical entities but also mathematical processes for relating, connecting, and operating on those entities (National Council of Teachers of Mathematics, 1989, 2000). A teacher must have such proficiency to set appropriate curricular goals for his or her students (Adler & Davis, 2006). For example, a teacher needs special mathematical knowledge to select and teach functions in a way that helps students build a basic repertoire of functions (Even, 1990).

Assess the Mathematical Knowledge of Learners

Assessing the mathematical knowledge of learners is an integral component of the mathematical work of teaching. During each class, teachers must exhibit a mathematical proficiency that enables them to assess or evaluate students' mathematical understanding. Such assessment is not only crucial for recognizing student error but also in determining where students are mathematically for purposes of developing tasks and planning lessons. Assessing students' mathematical knowledge involves much more than assessing a student's ability to follow a procedure. Teachers should possess a mathematical proficiency for teaching that helps them identify the essential components of mathematical concepts so they can in turn assess a student's ability to use and connect these essential ideas.

Examples of assessments that teachers frequently perform are:

1. A student might interpret the phrase "cancels out" to mean that the quotient of two equal terms is zero rather than one. Such confusion requires that teachers not only understand the mathematical error the student is making but also that they recognize that the source of the error may be located in how the student is using and understanding mathematical language.
2. Teachers also need to be able to choose sample problems that will probe student understanding during a lesson. For example, a teacher who wants to know if students are making the connection between the number of real-number solutions to a quadratic equation and its corresponding parabola may intentionally choose problems with no, one, and two solutions for students to solve and graph. During class discussion and by checking student work, the teacher can then assess whether the students are making this connection.
3. When going over work in class, a student might give the wrong answer to a problem because he or she is confused over the difference between the area and circumference of a circle. Such confusion may not be readily apparent to the teacher at the moment. Rather than quickly dismiss the answer, a teacher

who is proficient in the mathematical work of teaching will use open-ended questions to draw out the source of the student's confusion.

Determining how students are progressing in class is at the heart of assessing the mathematical knowledge of learners. To determine the mathematical progress of their students, teachers must be attentive to the errors students frequently make. Such errors include…

1. Confusing the use of area and perimeter,
2. Finding the reciprocal or multiplicative inverse of a function when asked to find its inverse,
3. Using values outside the domain of a function and
4. Confusing the use of sine and cosine.

Reflect on the Mathematics in One's Practice

The fifth category concerns proficiency in analyzing and reflecting on one's mathematics teaching practice in a way that enhances one's mathematical proficiency. There are many ways to reflect on one's practice, and one of the most important is to use a mathematical lens. How did the mathematical complexity of the problem in this lesson change when the students were given a hint? Which of several equivalent definitions is most appropriate when this term is introduced? How was the topic of that test question connected to a topic treated earlier in the course? Thoughtful reflection on problems of practice can be reconsideration of a lesson just taught, or it can be part of the planning for a future lesson. It may occur as the teacher interprets the results of a formal assessment, or it may be prompted by a textbook treatment of a topic.

Teachers are often reflecting about their teaching as they teach—as they are making split-second decisions. A teacher's decisions about how to proceed after accessing student thinking depend on many factors, including the mathematical goals of the lesson. It is valuable to revisit these quick reflections and decisions when there is time to think about the mathematics one might learn from one's practice.

Examples of reflection on the mathematics in one's practice are as follows:

1. Identifying an unconventional notation that students are using and contrasting its properties with those of conventional notation.
2. Analyzing how the topic of a lesson might be presented so as to show mathematics as culturally situated.
3. Modifying a mathematical conjecture so that it could be proved in other ways.

Background for the Framework

The evolution of this framework began with a desire to characterize mathematical knowledge for teaching at the secondary level. Our initial characterization was much influenced by the work of Deborah Ball and her colleagues at the University of Michigan (e.g., Ball, 2003; Ball & Sleep, 2007a, 2007b; Ball, Thames, & Phelps, 2008). In particular, Ball et al. have partitioned mathematical knowledge for teaching into components that distinguish between subject matter knowledge and pedagogical content knowledge (Shulman, 1986). As we worked on developing our own framework, we considered attempts to develop similar frameworks (e.g., Adler & Davis, 2006; Cuoco, 1996, 2001; Cuoco, Goldenberg, & Mark, 1996; Even, 1990; Ferrini-Mundy, Floden, McCrory, Burrill, & Sandow, 2005; McEwen & Bull, 1991; Peressini, Borko, Romagnano, Knuth, & Willis-Yorker, 2004; Tatto et al., 2008). Our intention has been to add to the work in this area, which continues to expand. We believe that our approach brings something new to the conversation about teachers' mathematical knowledge.

A new framework: Mathematical proficiency for teaching

Distinguishing MPT from MKT. Mathematical proficiency for teaching is related but not identical to mathematical knowledge for teaching (MKT). In examining the work that others have done in developing frameworks for MKT, we became increasingly convinced that whatever framework we developed should reflect a more dynamic view of mathematical knowledge. Therefore we have chosen to characterize mathematical *proficiency* rather than mathematical *knowledge*. Proficiency is the observable application of a teacher's knowledge and therefore reveals knowledge held by the teacher. Furthermore, MPT is related to, but different from, pedagogical content knowledge (Shulman, 1986). We focus on mathematics and do not attempt to describe pedagogical knowledge or proficiency.

Moving to the secondary school level. The framework presented here focuses exclusively on the mathematical proficiency that is beneficial to teachers of *secondary* school mathematics. We believe that MPT for secondary school is different from MPT for elementary school. Not only is there is a wider range of mathematics content at the secondary level, but there are also qualitative differences between elementary school mathematics and secondary school mathematics, and therefore a different kind of MPT is required. For example, in secondary school mathematics, mathematical proof requires a level of formality and rigor that is not necessary in elementary school. In addition the roles of mathematical structure and abstraction differ in a secondary classroom as compared with an elementary classroom. Finally, the mathematics that teachers and students encounter at the secondary level is unique in that students' cognitive abilities are developed to a point that they can engage in more advanced reasoning about mathematical concepts.

Starting from classroom practice. Our framework has been developed out of classroom practice, much like the work of Ball and her colleagues (e.g., Ball & Bass, 2003). A unique characteristic of our framework is the variety of classroom contexts from which we have drawn examples. We have observed the work of practicing teachers, preservice teachers, and mathematics educators and have used episodes from classrooms to examine and characterize MPT, as described in the following discussion of our development of *situations*.

Situations

Starting from the bottom up, we developed a collection of sample *situations* as a way of capturing classroom practice. Each situation portrays an incident in teaching secondary mathematics in which some mathematical point is at issue. (For details of our approach, see Kilpatrick, Blume, & Allen, 2006.) Looking across situations, we attempted to characterize the knowledge of mathematics that is beneficial for secondary school teachers to have but that other users of mathematics may not necessarily need.

Each situation begins with a *prompt*—an episode that has occurred in a mathematics classroom and raises issues that illuminate the mathematics proficiency that would be beneficial for secondary teachers. The prompt may be a question raised by a student, an interesting response by a student to a teacher's question, a student error, or some other stimulating event. We then outline, in descriptions called *mathematical foci*, mathematics that is relevant to the prompt. The set of foci is not meant to be an exhaustive accounting of the mathematics a teacher might draw upon, but we believe the foci include key points to be considered. These foci, each of which describes a different mathematical idea, constitute the bulk of each situation. There is no offer of pedagogical advice or comment about what mathematics the teacher should actually discuss in a class in which such an episode may occur. Rather, we describe the mathematics itself and leave it to the teacher or mathematics educator to decide what to use and how to do so. Along with the foci, each situation includes an opening paragraph, called a *commentary*, to set the stage for the mathematical foci. The commentary gives an overview of what is contained in the foci and serves as an advance organizer for the reader. Some situations also include a *post-commentary* to include extensions of the mathematics addressed in the situation.

Throughout the process of writing and revising the situations, we have used aspects of what we would come to include in our MPT framework. For example various *representations* helped us to think about the mathematics in the prompt. Perhaps there was a geometric model that was helpful or a graph or numerical representation to provide insight or clarification. At times a particular analogy was pertinent to the prompt. We were not interested in making every situation follow a particular format in which the same representations (such as analytical, graphical, verbal) were used again and again. We wanted to emphasize representations that we perceived as particularly helpful or relevant in relation to the prompt.

Another example of our use of aspects of mathematical proficiency in writing and revising situations was the use of *connections* to other mathematical ideas, or *extensions* to concepts beyond those currently at hand. For example, if a prompt addressed sums of integers, we described (though not in great detail) sums of squares. This is an example of a topic to be discussed in a post-commentary at the end of the situation. Another way to extend a mathematical focus is to adjust the assumptions. For example, in a geometry problem, one could consider the implications of relaxing the constraint of working only in Euclidean space.

Our use of these aspects of what would eventually constitute the MPT framework drew our attention to what we believed were pertinent elements of mathematical proficiency for teaching. This process helped us construct, clarify, and understand the framework and also provided us with examples to illustrate the elements of the framework.

Evolution of the Framework

By examining mathematical foci for about 50 situations, we developed a framework characterizing MPT for secondary school mathematics. In the situations, we could see the need, for example, for a teacher to be proficient in such tasks as using multiple representations of a mathematical concept, making connections between concepts, proving mathematical conjectures, determining the mathematics in a student comment or error, understanding the mathematics that comes before and after the task at hand, or discerning when students' questions raise mathematical issues that should be explored given the time available. In seeking to develop and improve the framework, we have responded to comments and suggestions given by experts (mathematicians, teacher educators, and teacher leaders [e.g. department heads]) in the field of mathematics education. We gathered this input at two Situations Development Conferences at the Pennsylvania State University, the first in May 2007 and the second in March 2009.

The purpose of the first conference was to present our work on ten of the situations to a group of mathematicians and mathematics educators. At that point we had not developed a framework; rather, we were at the stage of writing and revising situations with the goal of being able to characterize mathematical knowledge (later *proficiency*) for teaching at the secondary level and constructing a framework for doing so. We received input from the experts about the situations themselves, and that input challenged us to continue to refine our work and to consider some additional mathematical ideas that we had not included in the foci of the ten situations we shared. We also sought advice from the participants about what they considered to be key aspects of mathematical knowledge for teaching at the secondary level. A few of the ideas arising from that discussion were analysis of student thinking and student work, mathematical reasoning, mathematical connections, and mathematical habits of mind. We went back to work on the framework, trying to incorporate advice we had received at the conference, so as to continue the process of characterizing MKT (later MPT). We began to build lists of items (content and processes) to be included in the framework (e.g., entities such as mathematical connections and representations, and actions like choosing appropriate mathematical examples).

In March 2009, we presented a version of our framework to a group of mathematicians, mathematics educators, and teacher leaders for the purpose of seeking feedback and advice, as well as to discuss ideas about how the framework and situations could be used and disseminated. We received positive responses from participants regarding how they envisioned using the situations in their work with prospective or practicing teachers. The feedback we received on the framework document included comments about both the content and the format of the document. In small- and largegroup sessions, we had discussions about ideas for improving the framework—what to change or clarify, what to leave out, and what to add. Following the conference we began to work on incorporating these recommendations into our framework document.

At different times over the course of our work, we have focused on the situations, the framework, or both. Working on these two parts of our project in parallel has been helpful in keeping them both in view, particularly as our development of the situations has informed our construction of the framework. We believe that the framework now can be used to better interpret the situations, to write new situations, and to further our understanding of mathematical proficiency for teaching.

Conclusion

The work of describing MPT for secondary school mathematics continues, but we believe this framework is already an important contribution to the mathematics education community. Teachers require mathematical proficiency that is different from that needed in other professions. A teacher's work requires general mathematical knowledge as well as proficiency in the kinds of tasks described in this framework: accessing the mathematical thinking of learners, developing multiple representations of a mathematical concept, knowing how to use the curriculum in a way that will help further the mathematical understanding of students, and so on.

As we have said, a unique feature of our work is that we have chosen to develop a framework for mathematical *proficiency* for teaching (MPT), highlighting the dynamic nature of teacher knowledge. We believe that this focus is a valuable contribution to the field. That MPT is dynamic is one reason we have not arrived at a final formulation of MPT, and see this project as a work in progress. Mathematical proficiency for teaching should grow and deepen over the course of a teacher's career, and we expect our understanding of that proficiency to grow and deepen as well.

A second unique feature of the framework is that we focus solely on MPT at the secondary school level. We believe that this focus is essential to the profession's conversation about teacher knowledge. Secondary mathematics differs from elementary school mathematics in its breadth, rigor, abstraction, explicitness of mathematical structure, and level of reasoning required. Therefore, teaching at the secondary level requires a special kind of mathematical proficiency.

Finally, we believe we bring a unique perspective in that our framework has arisen from the practice of classroom teachers in a wide variety of settings including courses for prospective teachers, high school classes taught by practicing teachers, and classes taught by student teachers.

Just as we have sought the input of many mathematicians, mathematics teachers, and teacher educators during construction of this framework, we welcome comments from those in the field on our final product. Furthermore, we would like to gain further insight from others into MPT at the secondary level, perhaps by building on the ideas presented here.

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